NORM INEQUALITIES IN A CLASS OF FUNCTION SPACES INCLUDING WEIGHTED MORREY SPACES

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Abstract. We prove that Calderón-Zygmund operators, Marcinkiewicz operators, maximal operators associated to Bochner-Riesz operators, operators with rough kernel as well as commutators associated to these operators which are known to be bounded on weighted Morrey spaces under appropriate conditions, are bounded on a wide family of function spaces.

1. Introduction

In the last 10 years, many authors have been interested in norm inequalities involving classical and non-classical operators in the setting of weighted Morrey spaces. The n-dimensional Euclidean space $\mathbb{R}^n$ being equipped with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx, we call weight in $\mathbb{R}^n$ any positive measurable function w which is locally integrable on $\mathbb{R}^n$. Let w be a weight on $\mathbb{R}^n$, $1 \leq q < \infty$ and $0 < \kappa < 1$. The weighted Morrey space $L^{q,\kappa}_w(\mathbb{R}^n)$ consists of measurable functions f such that $\|f\|_{L^{q,\kappa}_w} < \infty$, where

$$\|f\|_{L^{q,\kappa}_w} := \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^q \, w(x)dx \right)^{\frac{1}{q}}. \tag{1.1}$$

These spaces which can be viewed as extensions of weighted Lebesgue space $L^q_w(\mathbb{R}^n)$, that is the spaces of Lebesgue measurable function f such that

$$\|f\|_{q_w} := \left(\int_{\mathbb{R}^n} |f(x)|^q \, w(x)dx \right)^{\frac{1}{q}} < \infty,$$

were first defined by Komori and Shirai in [14]. They generalize classical Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$, obtained by taking $w \equiv 1$ in (1.1) and $\lambda = n(1-\kappa)$.

For fixed $1 \leq q < \alpha$ and $1 \leq p < \infty$, Fofana introduced in 1992 a family of spaces denoted $(L^q, L^p)^\alpha$, which can be seen as intermediate spaces between the Lebesgue space $L^\alpha(\mathbb{R}^n)$ and the Morrey space $L^{q,\lambda}(\mathbb{R}^n)$ with $\lambda = \frac{nq}{\alpha}$. More precisely let $(L^q, L^p)(\mathbb{R}^n)$ denote the Wiener amalgam space of $L^q(\mathbb{R}^n)$ and

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$L^p(\mathbb{R}^n)$, i.e., the space consisting of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ which are locally in $L^q(\mathbb{R}^n)$ and such that the function $y \mapsto \left\| f\chi_{B(y,1)} \right\|_q$ belongs to $L^p(\mathbb{R}^n)$. Here, $B(y,r) = \{x \in \mathbb{R}^n / |x-y| < r\}$ is the open ball centered at y with radius r, $\chi_{B(y,r)}$ its characteristic function. Equipped with the norm defined by

$$\|f\|_{q,p} := \left(\int_{\mathbb{R}^n} \left\| f\chi_{B(y,1)} \right\|_q^p dy \right)^{\frac{1}{p}},$$

with the usual modification when $p = \infty$, is a Banach space. We refer the reader to the survey paper of Fournier and Steward [11] for more information about these spaces. It is easy to see that $\{\delta_r^\alpha f\}_{r>0} \subset (L^q, L^p)(\mathbb{R}^n)$ whenever $f \in (L^q, L^p)(\mathbb{R}^n)$, with $\delta_r^\alpha f(x) = r^{\frac{n}{\alpha}} f(rx)$, but the family is not bounded. In fact, for $r > 0$ and $\alpha > 0$, we have

$$\begin{aligned} \|\delta_r^\alpha f\|_{q,p} &= r^{n(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} \left(\int_{\mathbb{R}^n} \left\| f\chi_{B(y,r)} \right\|_q^p dy \right)^{\frac{1}{p}} \\ &\cong \left[\int_{\mathbb{R}^n} \left(|B(y,r)|^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{B(y,r)} \right\|_q \right)^p dy \right]^{\frac{1}{p}}, \end{aligned} \quad [1] \tag{1.2}$$

where $|B(y,r)|$ stands for the Lebesgue measure of $B(y,r)$ for $1 \leq q, p, \alpha \leq \infty$. This brings Fofana to consider the subspaces $(L^q, L^p)^\alpha(\mathbb{R}^n)$ of $(L^q, L^p)(\mathbb{R}^n)$ that consist in measurable functions f such that $\|f\|_{q,p,\alpha} < \infty$,

$$\|f\|_{q,p,\alpha} := \sup_{r>0} \|\delta_r^\alpha f\|_{q,p}.$$

Taking into consideration Relation (1.2), we generalized these spaces in the context of space of homogeneous type in the sense of Coifman and Weiss (see [9]).

It is proved in [10] that $(L^q, L^p)^\alpha(\mathbb{R}^n)$ spaces are nontrivial if and only if $q \leq \alpha \leq p$. In this case (see [2]) we have $(L^q, L^{p_1})^\alpha(\mathbb{R}^n) \subsetneq (L^q, L^{p_2})^\alpha(\mathbb{R}^n)$ for $\alpha \leq p_1 < p_2$. In fact, we have the following continuous injections

$$L^\alpha(\mathbb{R}^n) \hookrightarrow (L^q, L^{p_1})^\alpha(\mathbb{R}^n) \hookrightarrow (L^q, L^{p_2})^\alpha(\mathbb{R}^n) \hookrightarrow L^{q,\frac{nq}{\alpha}}(\mathbb{R}^n), \tag{1.3}$$

When $q < \alpha < p < \infty$, we can replace the Lebesgue space $L^\alpha(\mathbb{R}^n)$ in (1.3) with the weak Lebesgue space $L^{\alpha,\infty}(\mathbb{R}^n)$, and the inclusion is strict [10, 9].

We recall that a measurable function f belongs to the weak-Lebesgue space $L^{\alpha,\infty}(\mathbb{R}^n)$ if

$$\|f\|_{\alpha,\infty}^* := \sup_{\lambda>0} \lambda^{\frac{1}{\alpha}} |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| < \infty.$$

[1]Hereafter we propose the following abbreviation $\mathbf{A} \lesssim \mathbf{B}$ for the inequalities $\mathbf{A} \leq C\mathbf{B}$, where C is a positive constant independent of the main parameters. If we have $\mathbf{A} \lesssim \mathbf{B}$ and $\mathbf{B} \lesssim \mathbf{A}$ then we put $\mathbf{A} \cong \mathbf{B}$.

Let w be a weight on $\mathbb{R}^n$ and $1 \leq q, p, \alpha \leq \infty$. We define the space $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$ as the space of all measurable functions f satisfying $\|f\|_{q_w,p,\alpha} < \infty$, where for $r > 0$, we put

$$
{}_r\|f\|_{q_w,p,\alpha} := \left[\int_{\mathbb{R}^n} \left(w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|f\chi_{B(y,r)}\right\|_{q_w}\right)^p dy\right]^{\frac{1}{p}}, \tag{1.4}
$$

with $w(B(y,r)) = \int_{B(y,r)} w(x)dx$ and the usual modification when $p = \infty$, and

$$
\|f\|_{q_w,p,\alpha} := \sup_{r>0} {}_r\|f\|_{q_w,p,\alpha}.
$$

When $w \equiv 1$, we recover the space $(L^q, L^p)^\alpha(\mathbb{R}^n)$, while for $q < \alpha$ and $p = \infty$, the space $(L^q_w, L^\infty)^\alpha(\mathbb{R}^n)$ is noting but the weighted Morrey space $L^{q,\kappa}_w(\mathbb{R}^n)$, with $\kappa = \frac{1}{q} - \frac{1}{\alpha}$.

We prove in this paper that Calderón-Zygmund operators, Marcinkiewicz operators, maximal operators associated to Bochner-Riesz operators, operators with rough kernel and the commutators associated to these operators which are known to be bounded on weighted Morrey spaces under appropriate conditions, are also bounded on this weighted version of $(L^q, L^p)^\alpha(\mathbb{R}^n)$ spaces, under the same conditions. Since our space at least for the case where the weight is equal to 1, are included in Morrey spaces, we already know that the image is in the space of Morrey. But what is shown is that if one has a slightly stronger assumption, then this is also true for the image.

This paper is organized as follows.

In the next section, we recall the definitions of the operators we are going to deal with, and the existing results in weighted Lebesgue and Morrey spaces. Section 3 is devoted to the statement of our results and the last section to their proofs.

Throughout the paper, the letter C is used for non-negative constant independent of the relevant variables that may change from one occurrence to another. For $\lambda > 0$ and a ball $B \subset \mathbb{R}^n$, we write λB for the ball with same center as B and with radius λ times radius of B. We denote by E^c the complement of E.

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2. Definitions and known results

A weight w on $\mathbb{R}^n$ is of class $\mathcal{A}_q$ for $1 \leq q < \infty$ if there exists a constant $C > 0$ such that for all balls $B \subset \mathbb{R}^n$ we have

$$
\begin{cases} \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w^{\frac{-q'}{q}}(x)dx\right)^{\frac{q}{q'}} \leq C & \text{if} \quad q > 1, \\ \frac{1}{|B|}\int_B w(z)dz \leq C \operatorname{ess\,inf}_{x\in B} w(x) & \text{if} \quad q = 1. \end{cases} \tag{2.1}
$$

where and in what follows $\frac{1}{q}+\frac{1}{q'}=1$. We put $\mathcal{A}_\infty=\cup_{q\geq 1}\mathcal{A}_q$.

Let T be a Calderón-Zygmund operator given by

$$Tf(x)=\text{p.v}\int\limits_{\mathbb{R}^n}K(x-y)f(y)dy.$$

Here K is of class $\mathcal{C}^1(\mathbb{R}^n\setminus\{0\})$ and

$$|K(x)|\leq\frac{C_K}{|x|^n}\text{ and }|\nabla K(x)|\leq\frac{C_K}{|x|^{n+1}},\ x\neq 0.$$

The operator T is bounded on the weighted Lebesgue spaces L^q_w whenever $1<q<\infty$ and $w\in\mathcal{A}_q$, whereas for $q=1$ and $w\in\mathcal{A}_1$ we have the following weak type inequality

$$\|Tf\|^*_{q_w,\infty}:=\sup_{\lambda>0}\lambda w(\{x\in\mathbb{R}^n:|Tf(x)|>\lambda\}\lesssim\|f\|_{q_w}. \tag{2.2}$$

These results can be found in [6, 12]. In [14], Komori and Shirai extend them to weighted Morrey spaces $L^{q,\kappa}_w$.

Theorem 2.1 (Theorem 3.3 [14])**.** *If $1<q<\infty$, $0<\kappa<1$ and $w\in\mathcal{A}_q$, then T is bounded on $L^{q,\kappa}_w$.*

If $q=1$, $0<\kappa<1$ and $w\in\mathcal{A}_1$, then we have

$$\sup_{B:\text{ ball}}\frac{1}{w(B)^\kappa}\|(Tf)\chi_B\|^*_{q_w,\infty}\lesssim\|f\|_{L^{1,\kappa}_w}.$$

In the case $n\geq 2$, we denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. For any $\Omega\in L^\theta(\mathbb{S}^{n-1})$ with $1<\theta\leq\infty$, homogeneous of degree zero and such that

$$\int\limits_{\mathbb{S}^{n-1}}\Omega(x')d\sigma(x')=0,$$

where $x'=x/|x|$ for any $x\neq 0$, we define the homogeneous singular integral operator T_Ω by

$$T_\Omega f(x)=\text{p.v}\int\limits_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^n}f(x-y)dy,$$

and the Marcinkiewicz integral of higher dimension μ_Ω by

$$\mu_\Omega(f)(x)=\left(\int\limits_0^\infty\left|\int\limits_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

Duoandikoetxea in [6] proved that for $\Omega\in L^\theta(\mathbb{S}^{n-1})$ and $1<\theta<\infty$, if $\theta'\leq q<\infty$ and $w\in\mathcal{A}_{q/\theta'}$ then the operator T_Ω is bounded on $L^q_w(\mathbb{R}^n)$. One has the following in the weighted Morrey spaces.

Theorem 2.2 (Theorem 2 [19]). *Assume that $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta < \infty$. Then for every $\theta' \le q < \infty$, $w \in \mathcal{A}_{q/\theta'}$ and $0 < \kappa < 1$, there exists $C > 0$ independent of f such that*

$$\|T_\Omega f\|_{L_w^{q,\kappa}} \le C \|f\|_{L_w^{q,\kappa}}.$$

As far as Marcinkiewicz operators are concerned, it is proved in [3] that if $\Omega \in L^\theta(\mathbb{S}^{n-1})$ and $1 < \theta \le \infty$, then for every $\theta' < q < \infty$ and $w \in \mathcal{A}_{q/\theta'}$, there exists $C > 0$ such that

$$\|\mu_\Omega f\|_{q_w} \le C \|f\|_{q_w}. \tag{2.3}$$

The corresponding result in weighted Morrey spaces is stated as follows

Theorem 2.3 (Theorem 4 [19]). *Assume that $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta \le \infty$. Then for every $\theta' < q < \infty$, $w \in \mathcal{A}_{q/\theta'}$ and $0 < \kappa < 1$, there exists $C > 0$ independent of f such that*

$$\|\mu_\Omega f\|_{L_w^{q,\kappa}} \le C \|f\|_{L_w^{q,\kappa}}.$$

We also define the Bochner-Riesz operators of order $\delta > 0$ in terms of Fourier transforms by

$$(T_R^\delta f)\widehat{\ }(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ denote the Fourier transform of f. This operators can be expressed as convolution operators

$$T_R^\delta f(x) = (f * \phi_{1/R})(x), \tag{2.4}$$

where $\phi(x) = [(1 - |\cdot|^2)_+^\delta]\widehat{\ }(x)$. The associate maximal operator is defined by

$$T_*^\delta f(x) = \sup_{R>0} \left|T_R^\delta f(x)\right|.$$

Let $n \ge 2$. For $1 < q < \infty$ and $w \in \mathcal{A}_q$, Shi and Sun showed in [16] that $T_*^{(n-1)/2}$ is bounded on L_w^q. In the limit case $q = 1$ we have in [17] a weak type inequality when $w \in \mathcal{A}_1$, i.e.,

$$\left\|T_1^{(n-1)/2} f\right\|_{1_w}^* \lesssim \|f\|_{1_w}. \tag{2.5}$$

Putting together (2.5) and the fact that for a fixed $R > 0$

$$T_R^{(n-1)/2} f(x) = (\phi * f_R)_{1/R}(x),$$

this implies (see [18]) that the weak-type inequality (2.5) is satisfied for any $R > 0$.

Theorem 2.4 (Theorem 1 [18]). *Let $\delta = (n-1)/2$, $1 < q < \infty$, $0 < \kappa < 1$ and $w \in \mathcal{A}_q$. Then there exists $C > 0$ such that*

$$\left\|T_*^\delta (f)\right\|_{L_w^{q,\kappa}} \le C \|f\|_{L_w^{q,\kappa}}.$$

For the operators T_R^δ, we have the following in the context of Morrey spaces.

Theorem 2.5 (Theorem 2 [18]). *Let $\delta = (n-1)/2$, $q = 1$, $0 < \kappa < 1$ and $w \in \mathcal{A}_1$. Then for any given $R > 0$,*

$$\sup_{B:\ ball} w(B)^{-\kappa} \left\| T_R^\delta f \chi_B \right\|_{1_w,\infty}^* \lesssim \|f\|_{L_w^{1,\kappa}}.$$

We also have some results about commutators generated by those operators.

We recall that for a linear operator $\mathcal{T}$ and a locally integrable function b, the commutator operator is defined by

$$[b, \mathcal{T}]\, f(x) = b(x)\mathcal{T} f(x) - \mathcal{T}(bf)(x).$$

In [15], it is proved that for the Calderón-Zygmund operator T and $b \in BMO(\mathbb{R}^n)$, i.e., the space consisting of locally integrable functions satisfying $\|b\|_{BMO} < \infty$, where

$$\|b\|_{BMO} := \sup_{B:\ \text{ball}} \frac{1}{|B|} \int\limits_B |b(x) - b_B|\, dx,$$

with $b_B = \frac{1}{|B|}\int_B b(z)dz$, the commutators $[b, T]$ are bounded in the weighted Lebesgue space L_w^q whenever $1 < q < \infty$ and $w \in \mathcal{A}_q$. More precisely, there exists $C > 0$ such that

$$\|[b, T]\, f\|_{q_w} \leq C \|b\|_{BMO} \|f\|_{q_w},$$

for all $f \in L_w^q(\mathbb{R}^n)$. For the weighted Morrey spaces we have

Theorem 2.6 (Theorem 3.4 [14]). *Let $b \in BMO$ and T be a Calderón-Zygmund operator. If $1 < q < \infty$, $0 < \kappa < 1$ and $w \in \mathcal{A}_q$ then $[b, T]$ is bounded on $L_w^{q,\kappa}(\mathbb{R}^n)$.*

For $b \in BMO$, the boundedness of $[b, T_\Omega]$ on $L_w^q(\mathbb{R}^n)$ when $\Omega \in L^\theta(\mathbb{S}^{n-1})$, $1 < \theta < \infty$, $\theta' < q < \infty$ and $w \in \mathcal{A}_{q/\theta'}$ and the one of $\left[b, T_R^\delta\right]$ on $L_w^q(\mathbb{R}^n)$ for $1 < q < \infty$ and $w \in \mathcal{A}_q$ are just consequences of the well-known boundedness criterion for commutators of linear operators obtained by Alvarez et al in [1]. In the case of weighted Morrey, the following are proved.

Theorem 2.7 (Theorem 3 [19]). *Assume that $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta < \infty$ and $b \in BMO$. Then for every $\theta' < q < \infty$, $w \in \mathcal{A}_{q/\theta'}$ and $0 < \kappa < 1$, the commutator $[b, T_\Omega]$ is bounded on $L_w^{q,\kappa}(\mathbb{R}^n)$.*

Theorem 2.8 (Theorem 3 [18]). *Let $\delta \geq (n-1)/2$, $1 < q < \infty$, $0 < \kappa < 1$ and $w \in \mathcal{A}_q$. If $b \in BMO$ then for every $R > 0$, the operator $\left[b, T_R^\delta\right]$ is bounded on $L_w^{q,\kappa}(\mathbb{R}^n)$.*

We define the commutators of Marcinkiewicz operators μ_Ω and a locally integrable function b by

$$[b,\mu_\Omega](f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)dy\right|^2\frac{dt}{t^3}\right)^{\frac{1}{2}}.$$

Notice that $[b,\mu_\Omega](f)(x)=\mu_\Omega[(b(x)-b)f](x)$. For $\Omega\in L^\theta(\mathbb{S}^{n-1})$, $1<\theta\leq\infty$, the boundedness of $[b,\mu_\Omega]$ on L^q_w when $b\in BMO$, $\theta'<q<\infty$ and $w\in\mathcal{A}_{q/\theta'}$ was established in [3]. For weighted Morrey, Wang proved

Theorem 2.9 (Theorem 5 [19])**.** *Assume that $\Omega\in L^\theta(\mathbb{S}^{n-1})$ with $1<\theta\leq\infty$ and $b\in BMO$. Then for every $\theta'<q<\infty$, $w\in\mathcal{A}_{q/\theta'}$ and $0<\kappa<1$, the operator $[b,\mu_\Omega]$ is bounded on $L^{q,\kappa}_w(\mathbb{R}^n)$.*

3. Statement of the main results

We can replace in Relation (1.4) the weighted Lebesgue norm $\|\cdot\|_{q_w}$ by its weak version. That is, we put for $r>0$

$${}_r\|f\|_{(L^{q,\infty}_w,L^p)^\alpha}:=\left[\int_{\mathbb{R}^n}\left(w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\left\|f\chi_{B(y,r)}\right\|^*_{q_w,\infty}\right)^p dy\right]^{\frac{1}{p}},$$

and

$$\|f\|_{(L^{q,\infty}_w,L^p)^\alpha}:=\sup_{r>0}\ {}_r\|f\|_{(L^{q,\infty}_w,L^p)^\alpha}. \tag{3.1}$$

We denote $(L^{q,\infty}_w,L^p)^\alpha(\mathbb{R}^n)$ the space consisting of measurable functions f such that $\|f\|_{(L^{q,\infty}_w,L^p)^\alpha}<\infty$. Our first result is the boundedness of Calderón-Zygmund operators.

Theorem 3.1. *If $1<q\leq\alpha<p\leq\infty$ and $w\in\mathcal{A}_q$, then the Calderón-Zygmund operator T is bounded on $(L^q_w,L^p)^\alpha(\mathbb{R}^n)$.*

If $q=1$ and $w\in\mathcal{A}_1$, then we have

$$\|Tf\|_{(L^{1,\infty}_w,L^p)^\alpha}\lesssim\|f\|_{q_w,p,\alpha}.$$

Remark that this result contains Theorem 2.1 as particular case, while the next result giving the boundedness of the integral operator with a rough kernel Ω is an extension of Theorem 2.2.

Theorem 3.2. *Let $\Omega\in L^\theta(\mathbb{S}^{n-1})$ with $1<\theta<\infty$. Then for $\theta'\leq q\leq\alpha<p\leq\infty$ and $w\in\mathcal{A}_{q/\theta'}$, the operator T_Ω is bounded on $(L^q_w,L^p)^\alpha(\mathbb{R}^n)$.*

For Marcinkiewicz operators, we have the following extension of Theorem 2.3.

Theorem 3.3. *Let $\Omega \in L^\theta$ with $1 < \theta \leq \infty$. Then for $\theta' < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_{q/\theta'}$, the operator μ_Ω is bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.*

For Bochner-Riesz operators and their associated maximal functions, we recapitulate in the following the extensions of Theorems 2.4 and 2.5.

Theorem 3.4. *Let $1 \leq q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$.*

(1) If $q > 1$ and $\delta = \frac{n-1}{2}$, then T^δ_* is bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

(2) If $q = 1$ and $\delta = \frac{n-1}{2}$ then for any $R > 0$,

$$\left\|T^\delta_R f\right\|_{(L^{1,\infty}_w, L^p)^\alpha} \lesssim \|f\|_{q_w,p,\alpha}.$$

For commutators, we extend Theorems 2.6, 2.7, 2.8 and 2.9, and obtain the following results.

Theorem 3.5. *Le $b \in BMO$ and T be a Calderón-Zygmund operator. If $1 < q \leq \alpha < p \leq \infty$, and $w \in \mathcal{A}_q$ then the operator $[b, T]$ is bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.*

Theorem 3.6. *Let $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta < \infty$ and $b \in BMO(\mathbb{R}^n)$. For every $\theta' < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_{q/\theta'}$, the commutator $[b, T_\Omega]$ is bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.*

Theorem 3.7. *Let $\Omega \in L^\theta(\mathbb{S}^{n-1})$ with $1 < \theta < \infty$ and $b \in BMO(\mathbb{R}^n)$. For every $\theta' < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_{q/\theta'}$, the commutator $[b, \mu_\Omega]$ is bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.*

Theorem 3.8. *Let $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. If $\delta \geq \frac{n-1}{2}$ and $b \in BMO$ then the linear commutators $\left[b, T^\delta_R\right]$ are bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.*

4. Proof of the main results

We will need the following properties of $\mathcal{A}_q$ weights (see Proposition 9.1.5 and Theorem 9.2.2 [12]). Let $w \in \mathcal{A}_q$ for some $1 < q < \infty$. Then

(1) For all $\lambda > 1$ and all balls B, we have

$$w(\lambda B) \lesssim \lambda^{nq} w(B). \tag{4.1}$$

(2) There exists a positive constant γ such that we have

$$\left(\frac{1}{|B|}\int_B w(t)^{1+\gamma}dt\right)^{\frac{1}{1+\gamma}} \lesssim \frac{1}{|B|}\int_B w(t)dt, \text{ for all balls } B, \tag{4.2}$$

and for any measurable subset E of a ball B, we have

$$\frac{w(E)}{w(B)} \lesssim \left(\frac{|E|}{|B|}\right)^{\frac{\gamma}{1+\gamma}}. \tag{4.3}$$

Lemma 4.1. *Let $1 \leq s \leq q < \infty$, $w \in \mathcal{A}_{q/s}$ and $\mathcal{T}$ be a sublinear operator. Assume that there exist $0 \leq \eta < \infty$ such that $\mathcal{T}$ satisfies for any ball B in $\mathbb{R}^n$*

$$\mathcal{T}(f\chi_{(2B)^c})(x) \lesssim \sum_{k=1}^{\infty}(1+\eta k)\left(\frac{1}{|2^{k+1}B|}\int\limits_{2^{k+1}B}|f(z)|^s\,dz\right)^{\frac{1}{s}} \quad a.e.\ on\ B. \tag{4.4}$$

(1) *If $q > 1$ and $\mathcal{T}$ is bounded on $L^q_w(\mathbb{R}^n)$, then for $q \leq \alpha < p \leq \infty$, $\mathcal{T}$ is bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.*

(2) *If there exist $C > 0$ such that for all $\lambda > 0$*

$$w(\{x \in \mathbb{R}^n : |\mathcal{T}f(x)| > \lambda\} \leq \frac{C}{\lambda}\int\limits_{\mathbb{R}^n}|f(y)|\,w(y)dy, \tag{4.5}$$

then for $1 \leq \alpha < p \leq \infty$, $\mathcal{T}$ is bounded from $(L^1_w, L^p)^\alpha(\mathbb{R}^n)$ to $(L^{1,\infty}_w, L^p)^\alpha(\mathbb{R}^n)$.

For the proof we use arguments as in [7].

Proof. Let $1 < q \leq \alpha < p \leq \infty$. Fix $y \in \mathbb{R}^n$ and $r > 0$. Write

$$f(x) = f(x)\chi_{B(y,2r)}(x) + f(x)\chi_{(B(y,2r))^c}(x)$$

The L^q_w–boundedness of $\mathcal{T}$, (2.1) and (4.4) led to

$$\begin{aligned} \left\|\mathcal{T}f\chi_{B(y,r)}\right\|_{q_w} &\lesssim \left\|f\chi_{B(y,2r)}\right\|_{q_w} \\ &+ \textstyle\sum_{k=1}^{\infty}(1+\eta k)\left\|f\chi_{B(y,2^{k+1}r)}\right\|_{q_w}\left(\frac{w(B(y,r))}{w(B(y,2^{k+1}r))}\right)^{\frac{1}{q}}. \end{aligned} \tag{4.6}$$

Hence, multiplying both sides of (4.6) by $w(B)^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}$ we obtain

$$w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\left\|\mathcal{T}f\chi_{B(y,r)}\right\|_{q_w} \lesssim \sum_{k=0}^{\infty}\frac{1+\eta k}{2^{\frac{kn}{\tau'}(\frac{1}{\alpha}-\frac{1}{p})}}w(B(y,2^{k+1}r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_{q_w},$$

for all $y \in \mathbb{R}^n$ and a fixed positive real number τ, according to Relations (4.1) and (4.3). The L^p norm of the above estimation gives

$${}_r\left\|\mathcal{T}f\right\|_{q_w,p,\alpha} \lesssim (1+\sum_{k=1}^{\infty}\frac{1+\eta k}{2^{\frac{kn}{\tau'}(\frac{1}{\alpha}-\frac{1}{p})}})\left\|f\right\|_{q_w,p,\alpha},\ r > 0.$$

Taking the supremum over all $r > 0$, we obtain the expected result, since the series $\sum_{k=1}^{\infty}\frac{1+\eta k}{2^{\frac{kn}{\tau'}(\frac{1}{\alpha}-\frac{1}{p})}}$ converges.

As for the case $q = 1$, the proof is the same except for using Estimation (4.5) instead of the boundedness on L^q_w, Relation (4.4) corresponding to $q = 1$ and $w \in \mathcal{A}_1$. This completes the proof. □

The same arguments are used in [4] (see also [2]), to prove norm inequalities involving Riesz potentials and integral operators satisfying the hypothesis of Theorem 2.1 of [7] in the context of $(L^q, L^p)^\alpha(\mathbb{R}^n)$ spaces.

An immediate application of the above lemma is the following weighted version of Theorem 2.1 in [7].

Proposition 4.2. *Let $1 < q \le \alpha < p \le \infty$. Assume that $\mathcal{T}$ is a sublinear operator satisfying the property that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$*

$$|\mathcal{T}f(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy. \tag{4.7}$$

(1) If for $q > 1$ and $w \in \mathcal{A}_q$ the operator $\mathcal{T}$ is bounded on $L^q_w(\mathbb{R}^n)$ then it is also bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

(2) If for $w \in \mathcal{A}_1$ we have the weak type estimate

$$\|\mathcal{T}f\|^*_{1_w,\infty} \lesssim \|f\|_{1_w},$$

then we have

$$\|\mathcal{T}f\|_{(L^{1,\infty}_w, L^p)^\alpha} \lesssim \|f\|_{1_w,p,\alpha}.$$

Proof. Let $B(y,r)$ be a ball in $\mathbb{R}^n$. Notice that for $x, z \in \mathbb{R}^n$, we have

$$x \in B(y,r) \text{ and } z \notin B(y,2r) \Rightarrow |y-z| \le 2|z-x| \le 3|y-z|. \tag{4.8}$$

Thus for $x \in B(y,r)$, we have

$$\begin{aligned} \left|\mathcal{T}(f\chi_{(B(y,2r))^c})(x)\right| &\lesssim \int_{\mathbb{R}^n} \frac{\left|f\chi_{(B(y,2r))^c}(z)\right|}{|x-z|^n} dz \lesssim \sum_{k=1}^{\infty} \int_{2^k r \le |y-z| < 2^{k+1} r} \frac{|f(z)|}{|x-z|^n} dz \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{|B(y,2^{k+1}r)|} \int_{B(y,2^{k+1}r)\backslash B(y,2^k r)} |f(z)|\, dz. \end{aligned}$$

The conclusion follows from Lemma 4.1. □

The Proof of Theorem 3.1 is straightforward from Proposition 4.2.

For Theorems 3.2, 3.3 and 3.4, we prove that Hypothesis (4.4) of Lemma 4.1 is fulfilled to conclude.

Proof of Theorem 3.2. Let $B = B(y,r)$ be a ball of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we have

$$\left|T_\Omega(f\chi_{(2B)^c})(x)\right| \le \sum_{k=1}^{\infty} \left(\int_{2^{k+1}B\backslash 2^k B} \Omega((x-z)')^\theta dz \right)^{\frac{1}{\theta}} \left(\int_{2^{k+1}B\backslash 2^k B} \left(\frac{|f(z)|}{|x-z|^n}\right)^{\theta'} dy \right)^{\frac{1}{\theta'}}$$

by Hölder Inequality. From (4.8), it comes that for $x \in B$ and $z \in 2^{k+1}B\backslash 2^k B$, we have $2^{k-1}r \le |x-z| \le 2^{k+2}r$. Thus, for $x \in B(y,r)$ and any positive integer k, we have

$$\left(\int_{2^{k+1}B\backslash 2^k B} \Omega((x-z)')^\theta dz \right)^{\frac{1}{\theta}} \lesssim \|\Omega\|_{L^\theta(\mathbb{S}^{n-1})} \left|2^{k+1}B\right|^{\frac{1}{\theta}}, \tag{4.9}$$

and

$$\left(\int_{2^{k+1}B\setminus 2^kB} \left(\frac{|f(z)|}{|x-z|^n}\right)^{\theta'} dy\right)^{\frac{1}{\theta'}} \lesssim \frac{1}{|2^{k+1}B|}\left(\int_{2^{k+1}B} |f(z)|^{\theta'}\, dz\right)^{\frac{1}{\theta'}}. \tag{4.10}$$

Therefore, for any ball B in $\mathbb{R}^n$, we have

$$\left|T_\Omega(f\chi_{(2B)^c})(x)\right| \lesssim \sum_{k=1}^{\infty}\left(\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B} |f(z)|^{\theta'}\, dz\right)^{\frac{1}{\theta'}},$$

for all $x \in B$. Which ends the proof. □

Proof of Theorem 3.3. Put $g = f\chi_{(2B)^c}$ where B is a ball in $\mathbb{R}^n$. For $x \in B$ and $t > 0$ we have

$$\{z : |x-z| \leq t\} \cap (2^{k+1}B \setminus 2^kB) \neq \emptyset \Rightarrow t \geq 2^{k-1}r, \tag{4.11}$$

for any positive integer k. So

$$\begin{aligned}
|\mu_\Omega g(x)| &= \left(\int_0^\infty \left|\int_{(2B)^c\cap\{z:|x-z|\leq t\}} \frac{\Omega(x-z)}{|x-z|^{n-1}} f(z)dz\right|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}}\\
&\leq \sum_{k=1}^{\infty}\left(\int_{2^{k+1}B\setminus 2^kB} \frac{|\Omega(x-z)|}{|x-z|^{n-1}} f(z)dz\right)\left(\int_{2^{k-1}r}^\infty \frac{dt}{t^3}\right)^{\frac{1}{2}}\\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|^{1/n}}\int_{2^{k+1}B\setminus 2^kB} \frac{|\Omega(x-z)|}{|x-z|^{n-1}} f(z)dz,
\end{aligned}$$

for all $x \in B$. We end as in the proof of Theorem 3.2. □

Proof of Theorem 3.4. As in the above proof, we put $g = f\chi_{(2B)^c}$, where B is any ball in $\mathbb{R}^n$. Since for $R > 0$, $T_R^\delta(g)(x) = \left|g * \phi_{1/R}(x)\right|$ with $\phi(x) = \left[(1-|\cdot|^2)_+^\delta\right]\check{}(x)$, we have

$$\left|(g*\phi_{1/R})(x)\right| \leq R^n \int_{\mathbb{R}^n} \frac{|g(z)|}{(R\,|x-z|)^n}dz = \int_{(2B)^c} \frac{|f(z)|}{|x-z|^n}dz.$$

for $x \in B$, where we use the fact that $|\phi(x)| \lesssim \frac{1}{(1+|x|)^{\frac{n+1}{2}+\delta}}$ for $\delta \geq \frac{n-1}{2}$. The Assertions (1) and (2) follow from Proposition 4.2. □

For the results involving commutators, we need the following properties of BMO (see [13]). For $b \in BMO$, $1 < q < \infty$ and $w \in \mathcal{A}_\infty$ we have

$$\|b\|_{BMO} \cong \sup_{B:\text{ ball}} \left(\frac{1}{|B|} \int\limits_B |b(x) - b_B|^q \, dx \right)^{\frac{1}{q}}, \tag{4.12}$$

and for all balls B

$$\left(\frac{1}{w(B)} \int\limits_B |b(x) - b_B|^q \, w(x) dx \right)^{\frac{1}{q}} \lesssim \|b\|_{BMO} . \tag{4.13}$$

Let $b \in BMO$ and B a ball in $\mathbb{R}^n$. For all nonnegative integers k, we have

$$|b_{2^{k+1}B} - b_B| \lesssim (k+1) \|b\|_{BMO} . \tag{4.14}$$

Also, in almost all the proof, we use the following result which is just an application of Hölder Inequality, the properties of $\mathcal{A}_q$ weights and Estimation (4.13). The proof is omitted.

Lemma 4.3. *Let $1 \leq s < q < \infty$. For $b \in BMO$ and $w \in \mathcal{A}_{q/s}$, we have*

$$\left(\int\limits_{2B \backslash B} |b(z) - b_{2B}|^s \, |f(z)|^s \, dz \right)^{\frac{1}{s}} \lesssim \|b\|_{BMO} \, |2B|^{\frac{1}{s}} \, w(2B)^{-\frac{1}{q}} \, \|f\chi_{2B}\|_{q_w}$$

for all balls B and $f \in L^q_{loc}(\mathbb{R}^n)$.

Theorems 3.5, 3.6, 3.7 and 3.8 are immediate from the following weighted version of Theorem 2.2 in [7].

Proposition 4.4. *Let $1 \leq \theta < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_{q/\theta}$. Assume T is a sublinear operator which fulfilled conditions (4.4) with $s = \theta$ and admitted a commutator with any locally integrable function b, satisfying*

$$[b, T](f)(x) = T[(b(x) - b)f](x). \tag{4.15}$$

If $[b,T]$ is bounded on $L^q_w(\mathbb{R}^n)$, then $[b,T]$ is also bounded on $(L^q_w, L^p)^\alpha(\mathbb{R}^n)$.

Proof. Fix a ball $B = B(y, r)$ in $\mathbb{R}^n$. We have for all $x \in B(y, r)$

$$|[b,T](f)(x)| \lesssim |[b,T](f\chi_{2B})(x)| + |b(x) - b_B| \left|T(f\chi_{(2B)^c})(x)\right| + \left|T[(b_B - b)f\chi_{(2B)^c}](x)\right| .$$

Thus by the L^q_w-boundedness of $[b,T]$, (4.4) and (4.13), we have

$$\begin{aligned}\left\|[b,T]\,f\chi_B\right\|_{q_w} &\lesssim \left\|f\chi_{2B}\right\|_{q_w} + \|b\|_{BMO}\sum_{k=1}^{\infty}(1+\eta k)\left(\frac{w(B)}{w(2^{k+1}B)}\right)^{\frac{1}{q}}\left\|f\chi_{2^{k+1}B}\right\|_{q_w}\\ &+ w(B)^{\frac{1}{q}}\sum_{k=1}^{\infty}(1+\eta k)\left(\frac{1}{|2^{k+1}B|}\int\limits_{2^{k+1}B\backslash 2^kB}|b_B-b(z)|^{\theta}\,|f(z)|^{\theta}\,dz\right)^{\frac{1}{\theta}}.\end{aligned}$$

But then it comes from (4.14) and Lemma 4.3 that

$$\left(\int\limits_{2^{k+1}B}|b_B-b(z)|^{\theta}\,|f(z)|^{\theta}\,dz\right)^{\frac{1}{\theta}} \lesssim k\,\|b\|_{BMO}\,\frac{\left|2^{k+1}B\right|^{\frac{1}{\theta}}}{w(2^{k+1}B)^{\frac{1}{q}}}\left\|f\chi_{2^{k+1}B}\right\|_{q_w}.$$

Hence, we have

(4.16)

$$\begin{aligned}\left\|[b,T]\,f\chi_{B(y,r)}\right\|_{q_w} &\lesssim \left\|f\chi_{B(y,2r)}\right\|_{q_w}\\ &+ \|b\|_{BMO}\sum_{k=1}^{\infty}k^2\left(\frac{w(B(y,r))}{w(B(y,2^{k+1}r))}\right)^{\frac{1}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_{q_w},\end{aligned}$$

for all $y\in\mathbb{R}^n$. Therefore, multiplying both sides of (4.16) by $w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}}$, Estimates (4.1) and (4.3) and the L^p-norm allow to obtain

$$\text{(4.17)}\qquad {}_r\left\|[b,T]\,f\right\|_{q_w,p,\alpha} \lesssim \|b\|_{BMO}\,\Big(1+\sum_{k=1}^{\infty}\frac{k^2}{2^{\frac{nk}{\tau}\left(\frac{1}{\alpha}-\frac{1}{p}\right)}}\Big)\,\|f\|_{q_w,p,\alpha},$$

for some positive constant τ and all $r>0$. We end the proof by taking the supremum over all $r>0$. □

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